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A FORMULA FOR THE DENSITY OF SEA-WATER

B.Kullenberg
Oceanografiska institutionen, Göteborg

Observations of sigma-t for sea-water were published by R.A.Cox (the late), M.J.McCartney and F.Culkin (1970) together with an expression giving sigma as a function of salinity and temperature. The authors were primarily concerned with reporting new experimental data and did not aim at establishing a formula valid outside the range of the observations. Accordingly, their expression is not valid for fresh water and low salinities. The authors suggest that a formula taking into account the density of fresh water - as did, in fact, the formula established by Forch, Knudsen, Jacobsen and Sörensen - and the temperature of maximum density of sea-water would result in a more satisfactory function for sigma. The present author, having contemplated this idea for a long time, has used the new observations to establish such a function.

Denoting sigma-t for distilled water Σ_t, the maximum sigma-t for sea-water σ_m, and the temperature of maximum density t_m, the new formula was given the form

$$\sigma = \sigma_m + \Sigma_{t+3.9863-t_m} + c(t-t_m)^2 \qquad (1)$$

In order to facilitate the determination of the parameters t_m, σ_m and c as functions of S, the observed σ-values were recalculated to seven standard salinities, these being (with one exception) the salinities at which the largest number of sigma had been observed. As a rule the difference between the observed salinity and the "standard salinity" is smaller than 0.1 o/oo, but in some cases it exceeds 1.5 o/oo. The reduction of sigma-t was performed by means of the old formula in "Hydrographical Tables", the formula by Cox, McCartney and Culkin, and a preliminary formula established by the present author.

To determine the t_m giving the best fit for each (standard) salinity a number of closely spaced t_m-values were inserted in (1), then Σ was taken from L.W.Tilton and J.K.Taylor (1937) and σ_m and c were computed so as to make the sum of the squared residuals as small as possible for each t_m. A simple analytical expression was established for the sum of the squared residuals as a function of t_m, being of the form $a_o + a_1 t_m + a_2 t_m^2$, and the t_m-value making this sum a minimum was computed. The t_m-values arrived at are given in Table 1.

Table 1. Determination of t_m.

S	t_m	$\frac{S^2}{3.9863 - t_m}$	$(t_m)_{calc}$
9.579	2.123	49.24	2.0262
15.541	0.619	71.73	0.6952
20.128	-0.421	91.92	-0.3350
25.439	-1.615	115.53	-1.5266
29.698	-2.622	133.46	-2.4828
35.004	-3.565	162.26	-3.6742
39.760	-4.719	181.60	-4.7424

An analysis which it does not appear necessary to describe here has lead to the following analytical expression for t_m

$$t_m = 3.9863 - \frac{0.22473\ S^2}{S+0.941} \qquad (2)$$

It is, indeed, demonstrated by fig.1 that the quote $\frac{S^2}{3.9863 - t_m}$ is a linear function of S.

Using t_m-values determined by (2), given in Table 1 as $(t_m)_{calc}$, the σ_m- and c-values giving the best fit were determined and a simple expression for c was computed, making c = 0 at S = 0:

$$c \cdot 10^7 = -2.346\ S + 7.8112\ S^2 - 0.136398\ S^3 \qquad (3)$$

Finally the σ_m-values were computed which give the best fit when t_m and c are determined by (2) and (3); the results are given in Table 2.

Table 2. Determination of σ_m.

S	σ_m	$(\sigma_m)_{calc}$
9.579	7.7459	7.7448
15.541	12.4998	12.5078
20.128	16.1881	16.1835
25.439	20.4650	20.4606
29.698	23.9076	23.9098
35.004	28.2305	28.2329
39.760	32.1349	32.1336

It has been rather difficult to find an analytical expression for σ_m making σ_m=0 at S = 0. However, the following expression has been found to give a very good fit:

$$\sigma_m = 0.7737085\ S + 0.00059312\ S^2 + \frac{0.52553\ S}{S + 8.458} \qquad (4)$$

The σ_m-values determined by this expression are given in Table 2 as $(\sigma_m)_{calc}$.

Accordingly, the final formula for sigma reads as follows:

$$\sigma = \sigma_m + \Sigma_{t + 3.9863 - t_m} + c\ (t - t_m)^2 \qquad (5)$$

$$\Sigma_t = -\frac{(t - 3.9863)^2}{508.9292} \cdot \frac{t + 288.9414}{t + 68.12963}$$

where t_m, c and σ_m are determined by the expressions (2), (3) and (4) respectively.

Observed and calculated sigma-values are given in Table 3. The sum of the squared residuals for the 86 observations is 0.010792 giving a mean square deviation of 0.0113 and the corresponding figures using the formula published by Cox, McCartney and Culkin are almost identical, or 0.010815 and 0.0113 respectively. Both formulae are, therefore, equally accurate within the range of the observations, the difference between them being that the present formula is valid for fresh water and sea-water with a low salinity.

Further, a comparison has been made with the sigma-values published by Forch, Knudsen and Sörensen (1902), though only the fundamental observations given in "Tabelle a", page 83 in Knudsen's paper, "Bestimmung des spezifischen Gewichtes", have been used. The results are given in Table 4. One of the samples, No 20, has to be discarded, as the residual is about ten times the mean square deviation. The sum of the squared residuals of the remaining 44 observations is 0.006159, giving a mean square deviation of 0.0120. The mean of the residuals is -0.0063, and, therefore, a slight change of the expression for σ_m, so as to give the best fit for the Forch-Knudsen-Sörensen data, would for those data give a mean square deviation of 0.0102. This, as well as the observations made by Forch, will be further discussed in a paper to appear shortly. It would appear that the observations by Forch, Knudsen and Sörensen, published in 1902, are even better than has hitherto been assumed.

There is no doubt that it is possible to improve the expression giving sigma as a function of t and S. The present author will shortly publish an expression of the form

$$\sigma = \sigma_m - a \frac{(t-t_m)^2}{t+b} + c(t-t_m)^2$$

which, unfortunately, he was not able to finish before the deadline for sending in papers for this meeting.

Table 3. Observed values of sigma, salinity and temperature (Cox McCartney and Culkin), and values of sigma computed from the expression given by Cox, McCartney and Culkin, σ_{Cox}, and from expression (5) in this paper, σ_{Kg}.

$t^{o}C$	S	σ	σ_{Cox}	Residual	σ_{Kg}	Residual
0.000	40.288	32.401	32.4002	0.0008	32.3973	0.0037
	39.232	31.547	31.5478	-0.0008	31.5444	0.0026
	35.004	28.141	28.1377	0.0033	28.1340	0.0070
	29.698	23.878	23.8645	0.0135	23.8638	0.0142
	25.439	20.448	20.4397	0.0083	20.4429	0.0051
	20.154	16.194	16.1962	-0.0022	16.2035	-0.0095
	20.130	16.194	16.1770	0.0171	16.1843	0.0097
	15.541	12.492	12.4982	-0.0062	12.5040	-0.0120
	9.579	7.695	7.7268	-0.0318	7.7116	-0.0166
	40.288	31.906	31.9049	0.0011	31.8984	0.0076
	39.232	31.056	31.0674	-0.0014	31.0602	-0.0042
	35.004	27.708	27.7175	-0.0095	27.7086	-0.0006
	34.885	27.624	27.6232	0.0008	27.6144	0.0097
	29.698	23.514	23.5205	-0.0065	23.5129	0.0011
	25.439	20.157	20.1573	-0.0003	20.1538	0.0032
	20.128	15.981	15.9707	0.0103	15.9740	0.0070
	15.541	12.342	12.3613	-0.0193	12.3686	-0.0266
	9.896	7.944	7.9276	0.0164	7.9270	0.0170
	9.579	7.694	7.6789	0.0151	7.6770	0.0170
9.996	40.288	31.119	31.1182	0.0008	31.1143	0.0047
	39.232	30.277	30.2936	-0.0166	30.2892	-0.0122
	35.004	26.992	26.9956	-0.0036	26.9886	0.0035
	34.885	26.883	26.9028	-0.0198	26.8957	-0.0127
	29.698	22.864	22.8644	-0.0004	22.8557	0.0083
	25.439	19.551	19.5548	-0.0038	19.5475	0.0035
	20.128	15.445	15.4355	0.0095	15.4336	0.0115
	15.541	11.877	11.8849	-0.0079	11.8881	-0.0111
	9.896	7.539	7.5245	0.0145	7.5250	0.0140
	9.579	7.295	7.2799	0.0151	7.2796	0.0154
14.994	40.288	30.087	30.0759	0.0111	30.0783	0.0087
	39.232	29.268	29.2621	0.0059	29.2651	0.0029
	35.004	26.014	26.0078	0.0063	26.0094	0.0046
	34.885	25.931	25.9162	0.0148	25.9178	0.0132
	29.698	21.937	21.9321	0.0049	21.9296	0.0074
	25.439	18.667	18.6676	-0.0006	18.6634	0.0036
	20.128	14.614	14.6054	0.0086	14.6030	0.0110
	15.541	11.098	11.1048	-0.0068	11.1065	-0.0085
	9.579	6.582	6.5656	0.0164	6.5666	0.0154
17.494	41.390	30.308	30.3145	-0.0065	30.3169	-0.0089
	40.336	29.494	29.5065	-0.0125	29.5109	-0.0169
	40.288	29.510	29.4698	0.0403	29.4742	0.0358
	39.232	28.672	28.6606	0.0114	28.6665	0.0055
	39.230	28.663	28.6591	0.0039	28.6649	-0.0019
	35.027	25.433	25.4426	-0.0096	25.4489	-0.0159
	35.004	25.431	25.4250	0.0060	25.4313	-0.0003
	34.919	25.352	25.3600	-0.0080	25.3663	-0.0143

Table 3. continued

$t^{o}C$	S	σ	σ_{Cox}	Residual	σ_{Kg}	Residual
17.494	34.885	25.334	25.3340	-0.0000	25.3403	-0.0063
	29.723	21.376	21.3924	-0.0164	21.3941	-0.0181
	29.698	21.372	21.3733	-0.0013	21.3751	-0.0031
	29.729	21.384	21.3970	-0.0130	21.3987	-0.0147
	25.445	18.137	18.1329	0.0041	18.1314	0.0056
	25.439	18.118	18.1283	-0.0103	18.1269	-0.0089
	20.152	14.095	14.1090	-0.0140	14.1075	-0.0125
	20.128	14.081	14.0908	-0.0098	14.0893	-0.0083
	15.553	10.612	10.6208	-0.0088	10.6227	-0.0107
	15.541	10.604	10.6117	-0.0077	10.6136	-0.0096
	9.942	6.368	6.3754	-0.0074	6.3781	-0.0101
	9.629	6.123	6.1389	-0.0159	6.1411	-0.0181
	9.621	6.124	6.1328	-0.0088	6.1351	-0.0111
	9.579	6.118	6.1011	0.0169	6.1033	0.0147
19.993	41.379	29.648	29.6454	0.0026	29.6455	0.0025
	40.315	28.835	28.8339	0.0011	28.8375	-0.0025
	39.216	28.014	27.9961	0.0179	28.0023	0.0117
	35.017	24.799	24.7993	-0.0003	24.8087	-0.0097
	34.905	24.716	24.7141	0.0019	24.7235	-0.0075
	29.721	20.773	20.7765	-0.0035	20.7816	-0.0086
	29.710	20.774	20.7681	0.0059	20.7732	0.0008
	25.440	17.543	17.5322	0.0108	17.5329	0.0101
	20.147	13.523	13.5303	-0.0073	13.5293	-0.0063
	15.543	10.061	10.0578	0.0032	10.0594	0.0017
	9.940	5.841	5.8425	-0.0015	5.8459	-0.0049
	9.626	5.598	5.6066	-0.0086	5.6097	-0.0117
	9.618	5.598	5.6006	-0.0026	5.6037	-0.0057
24.992	41.398	28.181	28.2060	-0.0250	28.1835	-0.0025
	41.365	28.175	28.1810	-0.0060	28.1588	0.0163
	35.006	23.376	23.3780	-0.0020	23.3810	-0.0050
	34.890	23.293	23.2905	0.0025	23.2937	-0.0007
	29.712	19.386	19.3915	-0.0055	19.3935	-0.0075
	29.701	19.382	19.3833	-0.0013	19.3853	-0.0033
	25.436	16.189	16.1797	0.0093	16.1765	0.0125
	20.158	12.227	12.2254	0.0016	12.2180	0.0090
	20.156	12.226	12.2239	0.0021	12.2165	0.0095
	20.131	12.227	12.2052	0.0218	12.1978	0.0292
	15.536	8.781	8.7716	0.0094	8.7659	0.0151
	9.623	4.361	4.3658	-0.0048	4.3648	-0.0038
	9.621	4.363	4.3643	-0.0013	4.3633	-0.0003

Table 4. Observed values of sigma, salinity and temperature, σ_o for 0^oC and $\sigma_{24.6}$ for 24.6 ^oC, according to Forch, Knudsen and Sörensen (1902), and values of sigma computed from expression (5) in this paper. The salinity has been determined as S = 1.80655.Cl.
In the papers by Forch, Knudsen and Sörensen the water samples were allotted numbers which are given in the first column of this table; the sample marked "Schwedische" is here marked "S".

No	S	σ_o obs	σ_o calc	Res	$\sigma_{24.6}$ obs	$\sigma_{24.6}$ calc	Res
1	33.5914	26.9907	26.9961	-0.0054	22.4189	22.4333	-0.0144
2	35.0777	28.1777	28.1934	-0.0157	23.5310	23.5535	-0.0225
3	35.3922	28.4357	28.4468	-0.0111	23.7717	23.7904	-0.0187
4	35.3786	28.4310	28.4359	-0.0049	23.7680	23.7802	-0.0122
7	18.2769	14.6930	14.6983	-0.0053	10.9399	10.9219	0.0180
8	19.6430	15.7856	15.7938	-0.0082	11.9539	11.9438	0.0101
9	18.8075	15.1123	15.1238	-0.0115	11.3262	11.3187	0.0075
10	23.2001	18.6457	18.6465	-0.0008	14.6188	14.6103	0.0085
11	32.3298	25.9879	25.9805	0.0074	21.4821	21.4822	-0.0001
12	25.8254	20.7595	20.7531	0.0064	16.5899	16.5830	0.0069
16	32.7735	26.3304	26.3377	-0.0073	21.8020	21.8167	-0.0147
19	35.7034	28.7039	28.6977	0.0062	24.0281	24.0249	0.0032
20	36.4809	29.4394	29.3245	0.1149	24.7154	24.6105	0.1049
21	36.8798	29.6464	29.6462	0.0002	24.9065	24.9109	-0.0044
22	38.6721	31.0945	31.0923	0.0022	26.2667	26.2595	0.0072
23	40.1724	32.2927	32.3039	-0.0112	27.3874	27.3867	0.0007
25	28.9466	23.2570	23.2599	-0.0029	18.9276	18.9325	-0.0049
28	10.5454	8.4686	8.4902	-0.0216	5.1491	5.1574	-0.0083
29	8.3238	6.6828	6.6987	-0.0159	3.4900	3.5031	-0.0131
30	14.6128	11.7445	11.7591	-0.0146	8.1932	8.1866	0.0066
32	2.6622	2.0926	2.0857	0.0069	-0.7624	-0.7438	-0.0186
33	5.2885	4.2251	4.2376	-0.0125	1.2087	1.2365	-0.0278
S	33.9283	27.2546	27.2674	-0.0128	22.6687	22.6872	-0.0185

REFERENCES

Cox, R.A., M.J.McCartney and F. Culkin (1970) The specific gravity/ salinity/temperature relationship in natural sea water. Deep-Sea Research, vol.17, 679-689.

Forch C., M.Knudsen and S.P.Sörensen (1902) Berichte über die Konstantenbestimmungen zur Aufstellung der Hydrographischen Tabellen. Det K. danske Videnskabernes Selskabs Skrifter, 6. Raekke, Naturv. og Mat. Afd., 12, 1-151.

Tilton, L.W. and J.K.Taylor (1937) Accurate representation of the refractivity and density of distilled water as a function of temperature. J.Res.natn.Bur.Stand., 18, 205-214.

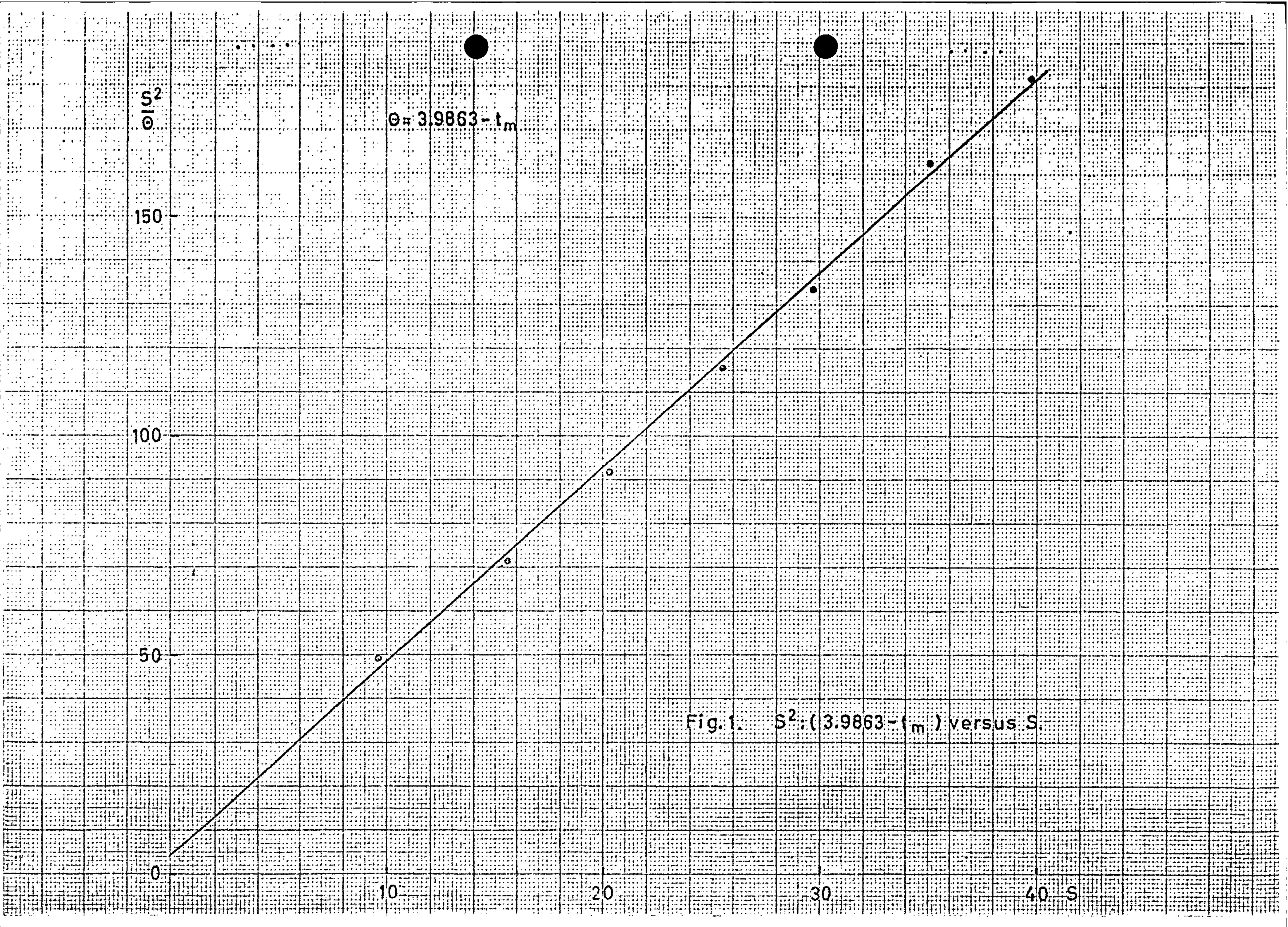

Fig. 1. $S^2 : (3.9863 - t_m)$ versus S.